A CONNECTED AND REGULAR POINT SET WHICH HAS NO SUBCONTINUUM*

BY

R. L. WILDER

It is well known† that every closed, connected and regular (connected im kleinen)‡ point set is arc-wise§ connected. That a connected and regular point set which is not closed is not necessarily arc-wise connected, and, indeed, may contain no arc, was shown by R. L. Moore.|| Since an arc is a very special kind of continuum,¶ Moore's result suggests the possibility of the existence of connected and regular point sets which not only fail to contain any arc, but do not contain any continuum whatsoever. It is the purpose of this paper to establish the existence of such sets.

I shall make use of an example of a connected set S containing a point a such that $S-a$ is totally disconnected, given by B. Knaster and C. Kuratowski.** For the sake of brevity, I shall assume familiarity with this set as well as with the proof of its connectivity.

I

Before proceeding to the construction of the point set which I wish to present in this paper, I shall first construct, and establish certain properties

---

* Presented to the Society, April 2, 1926; received by the editors in May, 1926.

† Cf. R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233–236; S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209; H. Tietze, *Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, Mathematische Zeitschrift, vol. 5 (1919), pp. 284–291.

‡ A point set M is said to be *regular* or *connected im kleinen at a point* P if for every circle K with center at P there exists a circle R concentric with K such that if x is a point of M interior to R, then P and x both lie in a connected subset of M which lies wholly interior to K. The set M is said to be *regular* or *connected im kleinen* if it is regular at every one of its points. Cf. H. Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1924), pp. 318–322. Also S. Mazurkiewicz, loc. cit.

§ A point set M is said to be *arc-wise connected* if every two of its points are the end points of at least one arc of M.

|| See Bulletin of the American Mathematical Society, vol. 29 (1923), p. 438.

¶ A *continuum* is a closed and connected point set which contains more than one point.

** *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255. The set S referred to here is described on pp. 241–244. Hereafter, I shall refer to this paper as S.E.C.

of, a set of which a certain important subset is in one-to-one continuous correspondence with the set $S-a$.

On the interval [0, 1] of the x-axis, let X be a non-dense perfect set. On all lines $x=\xi$, where ξ is the abscissa of an end point of an interval complementary to X, consider those points whose ordinates are rational numbers y such that $0\leqq y<1$. Call the set of all such points P_1. Let P_2 be the set consisting of all points whose ordinates are irrational numbers between 0 and 1 on lines $x=\xi$, where ξ is the abscissa of a point of X which is not an end point of an interval complementary to X.

There exists a one-to-one continuous correspondence between the set P_1+P_2 and the set $S-a$, which may be defined in the following manner: Let the set X described above and the set C as defined in §5 of S.E.C. be identical. For each point P of $S-a$ let x_P denote the abscissa of the point in which the x-axis is intersected by the line from P to a and let y_P denote twice the ordinate of P. There exists a correspondence between $S-a$ and P_1+P_2 in which to each point P of $S-a$ there corresponds the point of P_1+P_2 whose abscissa and ordinate are x_P and y_P respectively. This correspondence is continuous and will hereafter be called the correspondence Z.

As the segments* complementary to X form a denumerable set, they can be ordered in a sequence $e_1, e_2, e_3, \cdots$. The set of all points which lie above e_n ($n=1, 2, 3, \cdots$) and below $y=1$ denote by s_n. Let Q_j, $j=2^{n-1}$, $2^{n-1}+1, \cdots, 2^n-1$, be 2^{n-1} connected sets such that if L_i is the line $y=i/2^n$, where i takes on all odd integral values between 0 and 2^n, and P_{1in} and P_{2in} are the end points of that segment of L_i whose projection on the x-axis is e_n, then for $j=2^{n-1}+\frac{1}{2}(i-1)$, (1) Q_j contains P_{1in} and P_{2in}, is regular at these points, and, except for these two points, lies wholly in s_n; (2) all the limit points of Q_j except P_{1in} and P_{2in} lie in s_n; (3) if x is any point of Q_j, the distance of x from L_i is less than $1/2^n$.†

Let

$$M = P_1 + P_2 + \sum_{i=1}^{\infty} Q_i.$$

Definition. If A and B are two distinct points of a point set N, then A and B are *separated in N in the weak sense* provided N contains no connected subset which contains both A and B. The points A and B are *sepa-*

---

* By segment I mean an interval without its end points.

† This number is purely conventional, since all that is necessary for the proof below is that the diameter of Q_j approach zero as j increases.

rated in N in the strong sense if there exist two mutually separated* sets whose sum is N and which contain A and B, respectively.

The following lemma is part of a theorem which I have established in another connection, but which is not yet published:

LEMMA 1. *If N is a continuous curve,*† *and A and B are two distinct points of an open subset K of N which are not separated in K in the strong sense, then A and B are not separated in K in the weak sense.*

LEMMA 2. *Let J be a plane simple closed curve which is the sum of two arcs AxB and AyB which have only their end points A and B in common, R the bounded domain complementary to J, and C a continuum which separates the plane between A and B.*‡ *Then there exists a continuum C′ which is a subset of C and of J+R, and which has at least one point in common with each of the arcs AxB and AyB.*

Suppose no such continuum as C' exists. For every point P of C in R let $M(P)$ denote that maximal connected subset§ of $C\times(J+R)$ determined by P. Every set $M(P)$ has at least one point in common with one of the arcs AxB and AyB|| but not with both. Every point P whose corresponding $M(P)$ contains a point of AxB assign to a set M_1, and every point P whose corresponding $M(P)$ contains a point of AyB assign to a set M_2. There exists on AxB an arc $A'xB'$ which does not contain A or B, but which does contain every point of the set common to C and AxB. Similarly there exists on AyB an arc $A''yB''$ which does not contain A or B, but which contains every point common to C and AyB.

Let

$$A'xB' + M_1 = N_1,$$
$$A''yB'' + M_2 = N_2.$$

N_1 and N_2 are continua neither of which separates¶ A from B in $J+R$.

---

* Two point sets are said to be *mutually separated* if they are mutually exclusive and neither contains a limit point of the other.

† A *continuous curve* is a bounded, closed, connected and regular point set. A subset K of a continuous curve N is an *open subset* of N provided $N-K$ is a closed, non-vacuous set.

‡ I.e., there exists no connected subset in the plane which contains A and B and no point of C.

§ If P is a point of a point set M, then that maximal connected subset of M determined by P is the set of all points of M that lie with P in a connected subset of M.

|| Cf. Anna M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

¶ A subset N of a point set M is said to *separate* two points A and B of $M-N$, if $M-N$ is the sum of two mutually exclusive sets T and U which contain A and B respectively, and neither of which contains a limit point of the other.

Hence their sum, N_1+N_2, does not separate A from B in $J+R$.* That is, A and B are two points of the open subset $J+R-(N_1+N_2)$ of the continuous curve $J+R$, which are not separated in that open subset in the strong sense. Hence by Lemma 1, A and B are not separated in that open subset in the weak sense. Accordingly, there exists a connected subset of $J+R-(N_1+N_2)$ which contains A and B. As N_1+N_2 contains all points common to $J+R$ and C, a contradiction of the fact that C separates the plane between A and B results.

LEMMA 3. *The point set M described above is connected.*

Suppose M is not connected. Then it is the sum of two sets M_1 and M_2 which are mutually separated. Clearly no Q_i has points in both M_1 and M_2, as every Q_i is connected. Hence each of the sets M_1, M_2 contains points of the set P_1+P_2.

Let

$$M_1 \times (P_1 + P_2) = T_1,$$

$$M_2 \times (P_1 + P_2) = T_2.$$

T_1 and T_2 are mutually separated sets, since they are subsets of mutually separated sets, and

$$T_1 + T_2 = P_1 + P_2.$$

If m is a point of P_1+P_2 in T_1, say, then all those points of P_1+P_2 on the line $x=\xi_1$ which contains m belong to T_1. For suppose that q is a point of P_1+P_2 on $x = \xi_1$ belonging to T_2. Let the ordinate of q be less than that of m. By Theorem 37 (p. 233) of S.E.C., there exists a continuum C which contains no point of P_1+P_2 and which separates the plane between m and q.

Denote the intersection of $x=\xi_1$ and the x-axis by z. Then z is a limit point from the left, say, of end points of complementary intervals of X. Hence there exists an infinite sequence of distinct segments of the set $[e_n]$, viz., $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$, $\cdots$, whose end points have z as a sequential limit point,† and all of which lie to the left of z. Denote the corresponding elements of

* Cf. R. L. Moore, *Concerning upper semi-continuous collections of continua which do not separate a given continuum*, Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 356–360, Theorem 2.

† A point P is a sequential limit point of a point set K if every circle which encloses P also encloses all except a finite number of points of K.

the set $[s_n]$ by $\bar{s}_1, s_2, \bar{s}_3, \cdots$. Let b be a point whose abscissa is less than that of m, and whose ordinate is the same as that of m. Let d be a point which has the same abscissa as b and the same ordinate as q. Denote the interior of the rectangle $bmqdb$ by R. By Lemma 2, there exists a subcontinuum C' of C which contains points of the broken line $mbdq$ and of the straight line interval mq, and which, except for these points, is a subset of R.

Since m and q are not on C', and C' is closed, there exists a number $\eta<\xi_1$, such that if r and u are the intersections of $x=\eta$ with the straight line intervals bm and dq, respectively, there are no points of C' on the straight line intervals rm, uq. Also, there exist two elements $\bar{s}_i$ and $\bar{s}_j$ of the set $[\bar{s}_n]$ such that the end points of $\bar{e}_i$ and $\bar{e}_j$ lie between the point on the x-axis whose abscissa is η and the point z. Let (f) and (g) be abscissas of points in $\bar{e}_i$ and $\bar{e}_j$, respectively, and suppose $(f)<(g)$. There exists a subcontinuum $\bar{C}$ of C' which contains points of the lines $x=(f)$ and $x=(g)$, and which, except for these points, lies wholly between these two lines. As $\eta<(f)<(g)<\xi_1$, $\bar{C}$ lies wholly in R. Let the highest point of $\bar{C}$ on $x=(f)$ be f, and on $x=(g)$ be g, and let the projections of these two points on the x-axis be F and G, respectively. Let the smallest ordinate possessed by any point on $\bar{C}$ be ϵ, and let the intersections of $y=\epsilon$ with $x=(f)$ and $x=(g)$ be $\bar{f}$ and $\bar{g}$ respectively. Let those points of P_1+P_2 which lie within the rectangle $\bar{f}\bar{g}GF\bar{f}$ be denoted by $\bar{P}_1$. The set $\bar{P}_1$ is certainly not vacuous, for if ρ is the abscissa of the right-hand end point of $\bar{e}_i$, there are points of $\bar{P}_1$ on $x=\rho$. Let $\bar{P}_2$ denote the set of points of P_1+P_2 which lie between $x=(f)$ and $x=(g)$, and above $\bar{C}$.

Let w be any arc whose end points are F and G and which lies, except for F and G, entirely below the x-axis. Then the arc w, together with the straight line intervals fF and gG and the continuum $\bar{C}$, forms a continuum K which separates the plane between $\bar{P}_1$ and $\bar{P}_2$ and contains no point of P_1+P_2. Let H denote the set of all those points of P_1+P_2 which lie in that complementary domain of K which contains $\bar{P}_1$. The sets H and P_1+P_2-H are mutually separated. Let H^* and $(P_1+P_2-H)^*$ denote the subsets of $S-a$ which correspond to H and P_1+P_2-H respectively under the correspondence Z described above. It is easy to see that a is not a limit point of H. Hence $a+(P_1+P_2-H)^*$ and H^* are mutually separated. This is impossible, since S is connected. Thus the supposition that q is a point of T_2 leads to a contradiction and all points of P_1+P_2 on a line $x=\xi$ belong wholly to T_1, or to T_2, as the case may be.

Let the set of those points of X which are the projections of points of

$\tau_i (i=1, 2)$ on the x-axis be denoted by X_i. Then X_1 and X_2 are mutually separated.

If one end point of a segment e_n is in X_1, say, both end points are in X_1. Otherwise the point P_{11n} would be in T_1, say, and P_{21n} would be in T_2, which is impossible since Q_{2^n-1} lies wholly in one of the sets M_1, M_2. For every n, add all points of e_n to that set X_i to which its end points belong. Denote the sets obtained by adding points to X_1 and X_2 in this way by U_1 and U_2, respectively. Then U_1 and U_2 are mutually separated, since no point of a segment e_n is a limit point of points of the x-axis exterior to e_n. But this is impossible, since the set U_1+U_2 is identical with the set of all points in the interval [0, 1]. Hence the supposition that M is not connected leads to a contradiction.

LEMMA 4. *The point set M is regular at every point of the set P_1+P_2.*

Let t be a point of P_1+P_2 which does not belong to any Q_i. Let K_1 be any circle with center at t. Let W be a rectangle with center at t, lying wholly interior to K_1, with sides parallel to the coördinate axes and such that (1) the vertical sides of W contain no points of P_1+P_2, (2) if H is the height of W, the width of W is such that no elements of the set $[s_n]$ have points within W whose subscripts, n, do not satisfy the inequality $1/2^n < H/4$, it being understood that this condition does not apply to the elements of $[s_n]$ in which the vertical sides lie, (3) the upper and lower bases of W lie at a rational distance above the x-axis. Let K_2 be a circle concentric with K_1, lying interior to W, and not enclosing any points of those elements of the set $[Q_n]$ which have points on W. Such a circle exists, since t is not a point or a limit point of any Q_i and the diameters of the sets $[Q_n]$ approach zero as n increases. Clearly no point of P_1+P_2 interior to W is a limit point of the sum of all those sets of $[Q_n]$ which have points on W, unless it be a point of some such set.

Let N be the set of all points of P_1+P_2 interior to W or on the lower base of W, together with all sets Q_n which lie wholly interior to W. Clearly N is similar to the set M and it may be shown to be connected by the same argument that was used to prove M connected. As K_2 encloses only those points of M which belong to N, M is obviously regular at t.

If t is a point of P_1+P_2 belonging to a set Q_j, select W as before. Select K_2 so that it encloses no points of any set Q_n excepting Q_j, which has points on W. Select N as before. Since Q_j is regular at t, there exists a circle K_3, concentric with K_1, such that all points of Q_j interior to K_3 lie, with t, in a connected subset of Q_j which lies wholly interior to K_1. Then the smaller of the two circles K_2, K_3(or either of them, if they are identical) is clearly such that any point of M within it lies, with t, in a connected subset of M which lies wholly interior to K_1.

II

Consider a set N constructed as follows: On the interval $[0, 1]$ of the x-axis, let X be a non-dense perfect set. Denote the point $(\frac{1}{2}, 1)$ by P. For any point x of X, if x is an end point of an interval complementary to X, assign the straight line interval Px to a class E_1; otherwise to a class E_2. The set of all points on the xy-plane whose ordinates are rational and which lie on intervals of E_1 form a point set N_1, and the set of all points whose ordinates are irrational and which lie on intervals of E_2 form a set N_2. The set N_1+N_2 is connected, as it is identical, except for choice of coördinates, with the set S. Let the set N_1+N_2, together with the set of points symmetrical to it with respect to the x-axis, be denoted by Q_0. The set Q_0 is obviously connected. Let the point symmetrical to P with respect to the x-axis be denoted by P'.

As the intervals complementary to X on the interval $[0, 1]$ form a denumerable set, they can be ordered in a sequence $I_1, I_2, I_3, \cdots$. For every n, let e_{1n} and e_{2n} denote the end points of I_n, and let the set of points interior to the triangle $Pe_{1n}e_{2n}$ be denoted by s_n. Let Q_k^1, $k=2^{n-1}, 2^{n-1}+1, \cdots, 2^n-1$, be 2^{n-1} sets such that (1) for every k, Q_k^1 is in one-to-one continuous correspondence with Q_0, (2) if L_i is the line $y=i/2^n$, where i takes on all odd integral values between 0 and 2^n, and P_{1in} and P_{2in} are the intersections of $y=i/2^n$ with the straight line intervals Pe_{1n} and Pe_{2n}, respectively, then for $k=2^{n-1}+\frac{1}{2}(i-1)$, Q_k^1 (i) contains P_{1in} and P_{2in} and these points correspond, in the one-to-one continuous correspondence between this set and Q_0, to P and P', respectively, and (ii) is symmetrical with respect to the line $y=i/2^n$, has no limit points, except P_{1in} and P_{2in}, that do not lie in s_n, and if x is any point of it, the distance of x from L_i is less than $1/2^n$.

Denote the class each of whose elements is a set Q_i^1 $(i=1, 2, \cdots)$ by $[Q^1]^1$ and by $[Q^1]^2$ the class whose elements are symmetrical, with respect to the x-axis, to the elements of $[Q^1]^1$, and by $[Q^1]$ the class whose elements are the elements of $[Q^1]^1$ and $[Q^1]^2$.

Denote by $[Q^2]$ a class whose elements are constructed relative to *all* the elements of $[Q^1]$ just as the elements of $[Q^1]$ were constructed relative to Q_0. In general, denote by $[Q^n]$ a class whose elements are constructed relative to all the elements of $[Q^{n-1}]$ just as the elements of $[Q^1]$ were constructed relative to Q_0.

Let N be that point set which consists of all points of Q_0, as well as of all points which are contained in elements of the classes $[Q^i]$ $(i=1, 2, 3, \cdots)$.

THEOREM. *The point set N is a connected regular point set which contains no continuum.*

That N is connected is evident. That N is regular at P and P' is easily shown from the fact that Q_0 is itself regular at these points. Similarly, every element of $[Q^n]$ $(n=1, 2, 3, \cdots)$ is regular at the points corresponding to P and P'. The set $N-(P+P')$ is therefore a set of the same type as the set M described above, and is therefore regular at all points of the set $Q_0-(P+P')$ by Lemma 4.

For every n, arrange the elements of the class $[Q^n]$ in a sequence Q_1^n, Q_2^n, $Q_3^n, \cdots$. If Q_i^1 is any element of $[Q^1]$, it has been shown already that N is regular at the points of Q_i^1 corresponding to P and P'. As for the other points of Q_i^1, none of these is a limit point of the other elements of $[Q^1]$, and as Q_i^1 is a set of the same type as the set Q_0, N is regular at all these points. As for any element Q_i^n of any class $[Q^n]$, a similar argument holds, since no one of these sets contains, except for the points corresponding to P and P', a limit point of any element or set of elements of the classes $[Q^i]$ $(i=1, 2, 3, \cdots, n-1)$ or of $[Q^n]$ distinct from itself. Hence N is a regular point set.

The point set N contains no continuum. For suppose it does contain a continuum K. Then K contains a point, A, of an element Q_i^j of the class $[Q^j]$, say, where A is not one of the two points of Q_i^j corresponding to P and P' of Q_0. For the set of points in sets Q_i^j $(i=1, 2, \cdots, j=1, 2, \cdots)$ corresponding to P and P' of Q_0 is clearly denumerable and no continuum, nor indeed any connected set, can consist of a denumerable set of points. If d and f are the points of Q_i^j corresponding to P and P' in Q_0, it is easily shown by Theorem 1 of the paper of Miss Mullikin referred to above that K has a subcontinuum, K_0, which contains A, but does not contain d or f, nor any point of any other element of the class $[Q^j]$, nor any point of an element of a class $[Q^n]$ for which $n<j$. The continuum K_0 cannot lie wholly in Q_i^j since the latter set contains no continuum. Hence K_0 contains a point B of some set Q_k^m such that B is not a point of this set corresponding to P or P' in Q_0, and such that $m>j$. Then K_0 contains a subcontinuum K_1 which contains B but no point of an element of a class $[Q^n]$ for which $n<m$. By repetition of this argument, the existence of a sequence of continua

K_1, K_2, K_3, $\cdots$ may be shown such that for every positive integer q, K_{q+1} is a subset of K_q, and such that K_q contains no point of any element of a class $[Q^n]$ for which $n<q$. The sets K_1, K_2, K_3, $\cdots$ have in common at least one point P, and this point does not belong to any element of a set $[Q^n]$. Thus the supposition that N contains a continuum K is shown to lead to a contradiction, and the theorem is proved.

Ohio State University,
Columbus, Ohio